

Effects of Three-Dimensional Particle Orientation on Photoelastic Response in Polarization-Based Fluid Stress Measurement

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ABSTRACT

This study aims to establish a robust physical foundation for estimating internal fluid stress fields directly from polarization measurement images. Specifically, we systematically establish and reveal the complex relationship between internal fluid stress and polarization modulation within three-dimensional flow environments. To achieve this objective, we conducted a series of polarization measurement experiments using steady laminar flow through rectangular channels with different aspect ratios. The experimental data obtained from these measurements were compared with theoretical predictions estimated from two different theories: the classical stress-optic law (SOL), which traditionally correlates polarization modulation with fluid stress, and a theoretical model based on the Fokker-Planck equation, which incorporates particle orientation dynamics. Our comparative analysis revealed that the relationship between polarization modulation and internal fluid stress is dependent on the Peclet number Pe . Within the low Peclet number regime ($Pe < 1$), the experimental measurements demonstrated excellent agreement with the theoretical predictions of both the SOL and the Fokker-Planck models. However, as Pe progressively increased, the linear proportionality between optical response and fluid stress fundamentally broke down. This breakdown is attributed to the saturation of particle orientation. Consequently, in higher Pe regimes, the experimental values significantly deviated from the SOL predictions, whereas they maintained a strong general alignment with the predictions derived from the Fokker-Planck theory. Furthermore, the observed polarization modulation variations between channels with different aspect ratios highlighted an important theoretical need. We suggest that in the low Pe region, redefining Pe using a root-mean-square formulation at the channel center can effectively unify and identify the dependence on channel shape. Conversely, for regimes characterized by large Pe , our findings indicate that further in-depth investigations, strictly grounded in the fundamental dynamics of particle orientation, are essential. Ultimately, this provides the necessary framework for accurately interpreting the relationship between polarization modulation and fluid stress, particularly in scenarios where large distributions exist along the optical axis of the measurement system.

1. Introduction

Measurement of stress fields in fluids is important for understanding the physical mechanisms of fluid phenomena. For example, in the medical field, the stress acting on blood vessels is considered to be related to the rupture of cerebral aneurysms (Van Ooij et al., 2015). In general, fluid stress fields are estimated from the velocity field and its spatial gradients. However, conventional velocity measurement techniques, such as Particle Image Velocimetry (PIV) and hot-wire anemometry, still face several limitations, including insufficient spatial resolution and disturbance of the flow field caused by contact between the measurement device and the fluid.

To solve these issues, this study focuses on photoelastic measurement, one of the techniques used to measure stress in materials. This method enables non-contact stress measurement with high spatial and temporal resolution. Photoelastic measurement has traditionally been applied to the stress analysis of optically anisotropic solid materials. This method is based on the stress-optics law (SOL), which describes the relationship between stress and polarization modulation (the retardation Δ , the orientation ϕ) (Aben et al., 2000).

In contrast, fluids are generally optically isotropic. Therefore, in order to apply polarization-based measurements to fluids, it is necessary to introduce optical anisotropy into the fluid. To achieve this, optically anisotropic particles are dispersed in the fluid, and photoelastic measurements are performed by using the particle orientation induced by the flow field and fluid stress (Kusuno & Tagawa, 2025).

The polarization modulation measured in this case does not directly represent the local stress or particle orientation within the fluid, but rather is a two-dimensional image obtained as a result of integration along the camera's optical axis. Therefore, in order to obtain a three-dimensional stress field, it is necessary to reconstruct the three-dimensional stress field from a two-dimensional image. Several reconstruction methods have been proposed (Doyle & Danyluk, 1978; Dave et al., 2025), however, the validity of the reconstruction depends on the description of the process by which the three-dimensional stress field is observed as polarization modulation through integration in the camera's optical axis direction.

The conventional fluid polarization measurements have focused on the relationship between the macroscopic fluid stress field and polarization modulation. However, the modulation observed in a fluid occurs not through the stress field, but through particle orientation induced by the flow. Therefore, under high shear conditions, a particle orientation does not simply follow the macroscopic principal stress direction, which can lead to an apparent discrepancy between the polarization modulation amount and the stress field (Lane et al., 2022). These problems are thought to arise because polarization modulation does not directly reflect the fluid stress field, but rather through particle orientation determined by the flow field. Therefore, in order to estimate stress fields from polarization modulation, it is necessary to use particle orientation as a polarization

modulation variable and clarify what two-dimensional measurement quantities appear when its three-dimensional distribution is integrated along the camera's optical axis.

The relationship between polarization modulation and particle orientation has been studied as a "rheo-optic" response (Gravelle et al., 2021; Worby et al., 2024). In two-dimensional flows, the relationship between polarization modulation and particle orientation can be characterized by the Peclet number $Pe = |\dot{\gamma}| / Dr$, which represents the ratio between the orientation effect due to flow and the randomization effect due to Brownian motion (Gravelle et al., 2021; Yokoyama et al., 2026). Here, $\dot{\gamma}$ is the shear rate, and D_r is the rotational diffusion coefficient. When explaining the relationship between polarization modulation and particle orientation using the Peclet number Pe , it must be defined as the representative value integrated along the camera's optical axis. The two-dimensional flows that have been studied conventionally are flow fields that do not have a distribution in the direction of the camera's optical axis, and the Peclet number Pe is uniform even in the direction of the camera's optical axis. Therefore, it was relatively easy to correlate the measured change in modulation with a representative Peclet number Pe . However, in three-dimensional flows where the velocity distribution is non-uniform along the optical axis, the shear rate, particle orientation, and refractive index anisotropy have a non-uniform distribution. In this case, the polarization modulation amount measured by the camera is not a simple representative value of the local response along the optical path, but rather the result of integration along the optical path. Therefore, in a three-dimensional flow, the question arises as to which position of Pe governs the observed polarization modulation amount, or how to define the effective value of Pe distributed along the optical axis.

Therefore, this study aims to clarify the forward relation between the fluid flow, particle orientation, and polarization modulation, which is essential for stress reconstruction from polarization measurement images. To achieve this objective, polarization modulation in steady flow within a rectangular channel was experimentally measured and compared with predictions based on the conventional stress-optic law (SOL) and particle-orientation-based theory. Furthermore, the influence of the velocity distribution within the three-dimensional flow on the polarization modulation was investigated.

2. Theory

In this section, the calculation methods for the phase retardation Δ and the orientation ϕ are described. For the particle-orientation-based calculation, the orientation distribution function $\psi(t, \theta, \chi)$ was obtained by applying the Fokker-Planck equation (Talbot et al., 2024), which statistically accounts for the effects of Brownian motion, to Jeffery's orbit (Jeffery, 1922) describing the balance of rotational torque acting on particles in the flow. Here, t denotes time, while θ and χ represent the orientation angles of the particles. For the SOL-based calculation (Aben et al.,

2000; Doyle & Danyluk, 1978), the phase retardation and orientation angle were evaluated using a stress-optic law that includes the stress component along the camera optical axis as a second-order term.

2.1. Jeffery's orbit

Jeffery's orbit is an equation that describes the motion of a single rod-shaped particle in the velocity field $\mathbf{u}$ of a Stokes flow. This equation is based on the balance between the rotational torque the particle receives from the fluid and the geometric constraints on the particle, and express as (Jeffery, 1922; Talbot et al., 2024)

$$\dot{\mathbf{q}} = \mathbf{W} \cdot \mathbf{q} + \beta \left[\mathbf{E} \cdot \mathbf{q} - (\mathbf{q}^T \cdot \mathbf{E} \cdot \mathbf{q}) \mathbf{q} \right], \quad (1)$$

where $\mathbf{q}$ is a unit vector specifying the particle orientation, $\mathbf{W}$ is the vorticity tensor ($\mathbf{W} = (\nabla \mathbf{u} - \nabla \mathbf{u}^T) / 2$) and $\mathbf{E}$ is the strain rate tensor ($\mathbf{E} = (\nabla \mathbf{u} + \nabla \mathbf{u}^T) / 2$). β is the shape factor

$$\beta = \frac{(l/d)^2 - 1}{(l/d)^2 + 1}, \quad (2)$$

where l and d are the length and diameter of the rod particles. Furthermore, $\mathbf{q}$ is defined using the spherical coordinate system of the unit sphere as follows

$$\mathbf{q} = \begin{pmatrix} q_x \\ q_y \\ q_z \end{pmatrix} = \begin{pmatrix} \sin \theta \cos \chi \\ \sin \theta \sin \chi \\ \cos \theta \end{pmatrix} \quad (0 \leq \theta < \pi, 0 \leq \chi < \frac{\pi}{2}). \quad (3)$$

2.2. Fokker-Planck equation

Jeffery's orbit considered the hydrodynamic torque exerted on particles by the flow, but particle orientation is also influenced by the Brownian motion of the particles themselves. Therefore, we considered the effect of Brownian motion by using the orientation distribution function ψ of particle orientation with the Fokker-Planck equation (Talbot et al., 2024)

$$\frac{\partial \psi}{\partial t} = \nabla \cdot (D_r \nabla \psi) - \nabla \cdot (\dot{\mathbf{\Omega}} \psi). \quad (4)$$

Here, $\mathbf{\Omega} = (\chi, \theta)^T$, which can be obtained from Eq. (1). D_r is the rotational diffusion coefficient and mainly depends on the rod length l as

$$D_r = \frac{3k_B T \ln(l/d)}{\pi \eta l^3}, \quad (5)$$

where k_B is the Boltzmann constant, T is the absolute temperature, and η is the solvent viscosity. The order parameter S , which represents the degree of alignment of particle orientation in the xy plane, is (Gravelle et al., 2021)

$$S = \sqrt{\left(\langle q_x^2 \rangle - \langle q_y^2 \rangle\right)^2 + 4 \langle q_x q_y \rangle^2}, \quad (6)$$

where $\langle - \rangle$ represents the ensemble mean with respect to the orientation distribution function ψ . When $S = 1$, the particle orientation is perfectly uniform, and when $S = 0$, the particle orientation is completely random.

2.3. Particle orientation and polarization

The retardation Δ caused by particle orientation is the integrated amount of birefringence on the optical axis resulting from the alignment of particle orientations, and is expressed by the order parameter S as follows (Gravelle et al., 2021)

$$\Delta = S \Delta_{\max}. \quad (7)$$

Here, $\Delta_{\max}$ is the retardation when the particle orientation is perfectly aligned ($S = 1$), and is expressed as

$$\Delta_{\max} = 2hc\Delta n_0, \quad (8)$$

where c is the volume fraction of the particle, Δn_0 is the intrinsic birefringence of the particle, and $2h$ represents the integrated distance. Furthermore, the orientation ϕ is correlated with the direction the particle is facing, and it has been suggested that (Gravelle et al., 2021)

$$\phi = \frac{1}{2} \tan^{-1} \frac{\langle 2q_x q_y \rangle}{\langle q_x^2 - q_y^2 \rangle}. \quad (9)$$

2.4. Stress-optic law

The stress-optics law (SOL) is a theory that describes the relationship between stress and polarization modulation in a material. The polarization modulation (phase difference Δ , orientation ϕ) obtained by SOL is the integral along the direction of the camera's optical axis (Aben et al., 2000)(Doyle & Danyluk, 1978). When applying SOL to a fluid, the following relationship, which includes the stress in the direction of the camera's optical axis as a second-order term (Nakamine et al., 2024)(Worby et al., 2024):

$$V_1 = \Delta \cos 2\phi = \int \left\{ C_1 (\sigma_{xx} - \sigma_{yy}) + C_2 \left[(\sigma_{yy} + \sigma_{xx}) (\sigma_{yy} - \sigma_{xx}) + \sigma_{yz}^2 - \sigma_{xz}^2 \right] \right\} dz, \quad (10)$$

$$V_2 = \Delta \sin 2\phi = \int \{2C_1\sigma_{xy} + C_2[2(\sigma_{xx} + \sigma_{yy})\sigma_{xy} + 2\sigma_{xz}\sigma_{yz}]\} dz, \quad (11)$$

where σ_{ij} is the fluid stress component, z is the camera's optical axis direction, and C_1 and C_2 are the stress optical coefficients. In solids, $C_2 = 0$ can be considered, but in fluids, if there is a stress component along the camera's optical axis, then $C_2 \neq 0$ (Worby et al., 2024).

Therefore, the retardation Δ and the orientation ϕ are given by

$$\Delta = \sqrt{V_1^2 + V_2^2} \quad (12)$$

$$\phi = \frac{1}{2} \tan^{-1} \frac{V_2}{V_1}. \quad (13)$$

3. Methodology

3.1. Working fluid

In this study, optical anisotropy was imparted to a fluid by dispersing cellulose nanocrystals (CNCs), which are particles with a high aspect ratio. To make it easier to observe the orientation response of CNCs under flow conditions by increasing the viscosity of the dispersion, a glycerol/water mixed solvent (glycerol 80 wt%) was used as the dispersion medium, and the concentration of CNCs were adjusted to 0.05 wt%.

The viscosity of the prepared working fluid is shown in Fig. 2. The viscosity measurements were performed using a stress-controlled rheometer (MCR302, Anton Paar) with a parallel cone plate. Measurements were taken for shear rates $\dot{\gamma}$ in the range of 1 to 1000 s⁻¹. From Fig. 2, it was observed that there was almost no dependence on shear rate within the measured range of shear rates. Therefore, in this study, it was treated as a Newtonian fluid. The viscosity of each dispersion was defined as the average value in the range $\dot{\gamma} = 1\text{--}1000\text{s}^{-1}$, and these values were summarized in Table 1. Because the experiment was conducted on different days for each channel, the room temperature and viscosity values varied.

The particle shape distribution (When the particle is considered as a cylinder, the length of the particle is l and the diameter of the particle is d .) of CNC particles measured by an atomic force microscope (AFM1000 NanoView, Suzhou FlyingMan Precision Instruments) is shown in Fig. 1. From Fig. 1, it can be seen that the CNCs are polydisperse in both length and diameter. Since the polarization modulation depends on the shape of the particles, polydispersity must be taken into consideration. In this study, following Yokoyama et al. (Yokoyama et al., 2026), we estimated the particle parameter distribution by fitting the particle shape distribution to a moment-weighted

log-normal family

$$f_n(x) = \frac{x^n f_0(x)}{\int_0^\infty x^n f_0(x) dx} = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\ln x - (\mu + n\sigma^2))^2}{2\sigma^2}\right), \quad (14)$$

where μ is the logarithm of the mean, σ is the logarithm of the geometric standard deviation, $f_0(x)$ is the log-normal distribution, and n is the weighting parameter. In the actual calculations, μ and σ were used as the sample mean and sample standard deviation, respectively. $f_0(x)$ is a log-normal family representing the distribution of particle shapes, and is given by:

$$f_0(x) = \frac{1}{\sqrt{2\pi}\sigma x} \exp\left(-\frac{(\ln x - \mu)^2}{2\sigma^2}\right). \quad (15)$$

Ultimately, the particle shape can be estimated using a moment-weighted log-normal family $f_n(x)$ as follows:

$$\begin{aligned} \langle x \rangle &= \int_0^\infty x f_n(x) dx \\ &= \exp\left(\mu + \frac{(2n+1)}{2}\sigma^2\right). \end{aligned} \quad (16)$$

Yokoyama et al. suggested that by modeling the moment order as $n = 2$, a theoretical model that considers the particle shape distribution can more appropriately reproduce experimental results (Yokoyama et al., 2026). Therefore, in this study, we modeled the moment order as $n = 2$, obtaining $\langle l \rangle = 286$ nm and $\langle d \rangle = 7.3$ nm.

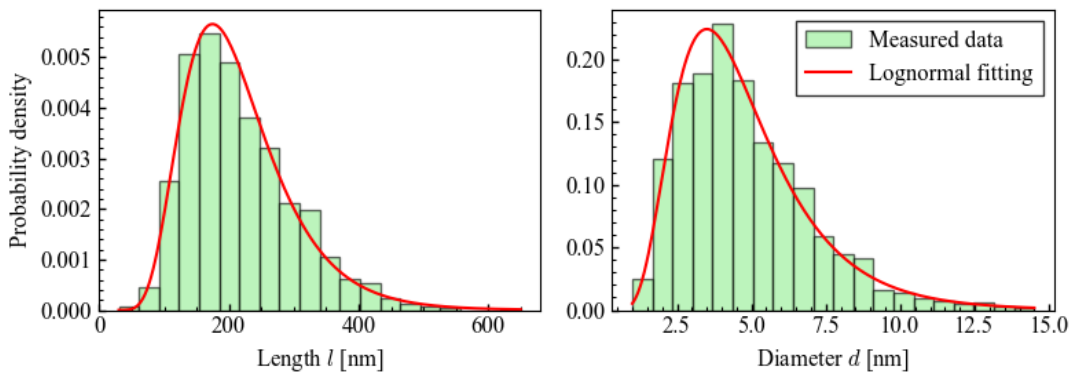


Figure 1. Distribution of particle length l (left) and particle diameter d (right).

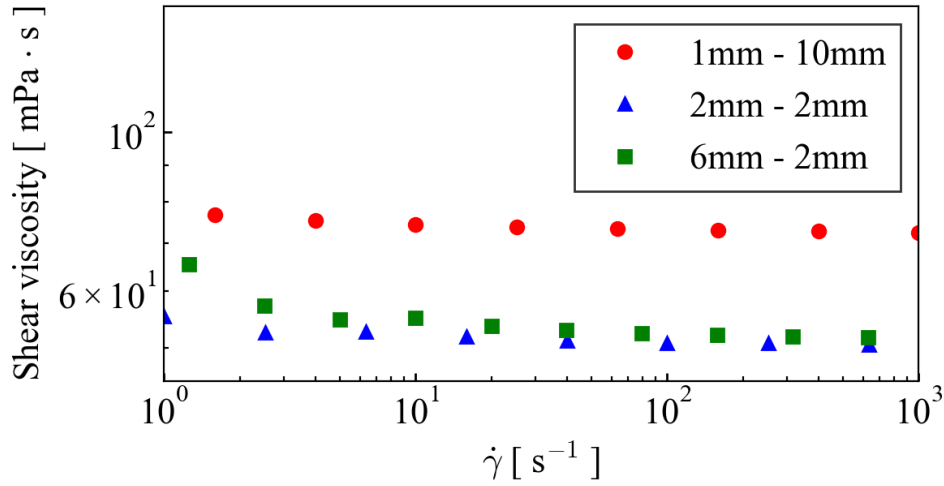


Figure 2. Shear viscosity of the working fluid.

Table 1. Properties of the working fluid.

Label	Temperature (°C)	η (mPa·s)
1mm–10mm	17	73.3
2mm–2mm	21	51.3
6mm–2mm	23	51.9

3.2. Experiment setup

A schematic diagram of the experimental setup is shown in Fig. 3, along with the coordinates of the system. A rectangular channel was placed between a light source (SOLIS-565C, Thorlabs) that generates light with a typical wavelength of 540 nm and a high-speed polarization camera (CRYSTA PI-5WP, Photonic Lattice, imaging speed: 200 f.p.s.). The light passes through the linear polarizer and the quarter-wave plate and becomes circularly polarized. Circularly polarized light is modulated as it passes through a channel, influenced by the particle orientation within the fluid. Polarized-modulated light is measured by a polarization camera as a retardation Δ and an orientation ϕ . In this study, the flow direction was defined as the x -axis, the camera's optical axis as the z -axis, and the direction perpendicular to the camera's optical axis as the y -axis. The fluid was supplied using a syringe pump (PHD ULTRA CP, HARVARD Apparatus) to produce a steady laminar flow. The flow rate Q was varied in increments of 10 ml/min within the range of 10 – 100 ml/min. We prepared three different aspect ratio channels ($w/h = 0.1, 1, 3$). The aspect ratio $w/h = 0.1$ was produced using a channel width of $2w = 1$ mm and a channel depth of $2h = 10$ mm. The aspect ratio $w/h = 1$ was produced using a channel width of $2w = 2$ mm and a channel depth of $2h = 2$ mm. The aspect ratio $w/h = 3$ was produced using a channel width of $2w = 6$ mm and a channel depth of $2h = 2$ mm. The channel with $w/h = 1$ was made of quartz glass. The channels with aspect ratios $w/h = 0.1$ and 3 were fabricated using a stereolithography 3D

printer (Form3, Formlabs, material: clear resin). To ensure the optical path formed a flat surface, the channels were sealed by sandwiching their front and back sides between two glass microscope slides.

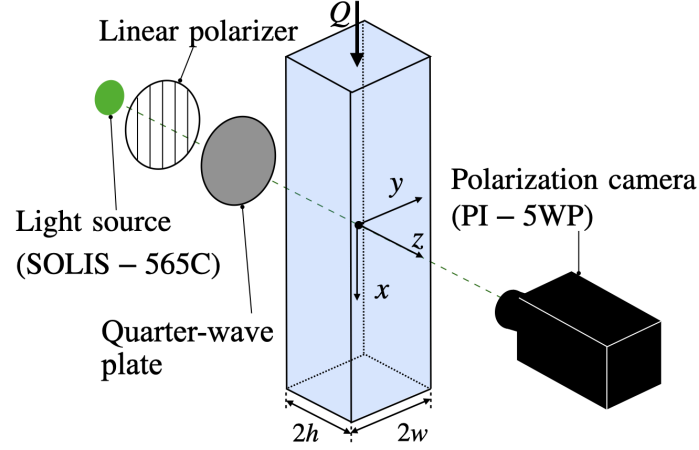


Figure 3. A schematic of photoelastic measurement setup.

3.3. Theoretical flow

This section describes the theoretical flow field within a rectangular channel. The velocity distribution of steady laminar flow in a rectangular channel is expressed by the Navier-Stokes equations as follows (Delplace, 2018)(Nagamoto et al., 2006)

$$\left(\frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) u_x(y, z) = -\frac{4Q}{whK}. \quad (17)$$

Here, $u_x(y, z)$ is the flow velocity in the x -direction and K is

$$K = \frac{16}{3} - \frac{1024}{\pi^5} \frac{h}{w} \sum_{n=0}^{\infty} \frac{1}{(2n+1)^5} \tanh \frac{(2n+1)\pi w}{2h}. \quad (18)$$

By imposing a no-slip boundary condition on Eq. (17), we can obtain the velocity distribution of $u_x(y, z)$

$$u_x(y, z) = \frac{16Q}{\pi^3 whK} \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)^3} \left[1 - \frac{\cosh \frac{(2n+1)\pi z}{2h}}{\cosh \frac{(2n+1)\pi w}{2h}} \right] \cos \frac{(2n+1)\pi y}{2w}. \quad (19)$$

In this study, we proceed with the assumption that the velocity distribution in the experiment is Eq. (19).

4. Results / Discussions

4.1. Polarization measurement results

The result of polarization measurement is shown in Fig. 4 for a channel with $w/h = 1$ and a flow rate of 60 ml/min. The left side shows the measured retardation Δ , and the right side shows the measured orientation ϕ , with the flow from top to bottom. Please note that the measured information is the result of integrating the distributions of the retardation Δ and orientation ϕ in the direction of the camera's optical axis (the z axis). The measured polarization modulation is uniform in the x -direction, indicating that the flow is well-developed.

Fig. 5 shows line profiles of the polarization modulation under different flow rate Q for each channel geometry ((a): $w/h = 10$, (b): $w/h = 1$, and (c): $w/h = 3$). The line profiles represent the average amount of polarization modulation in both the temporal direction and the x -direction. The retardation Δ increased with increasing flow rate Q . For all conditions, the largest retardation was observed at the channel walls, whereas the smallest retardation was observed at the channel center. The magnitude of the orientation ϕ decreased as the flow rate Q increased. In all cases, the largest magnitude of ϕ was observed on the channel walls, and it gradually decreased toward the channel center, where $\phi = 0$. Since the flow field in the rectangular channel is expected to be symmetric with respect to the channel centerline, the measured polarization modulation was also expected to exhibit a symmetric distribution. However, slight asymmetries were observed in some of the experimental results. These asymmetries are considered to originate from channel fabrication errors, imperfect alignment between the channel and the camera, and measurement uncertainties associated with the camera system itself.

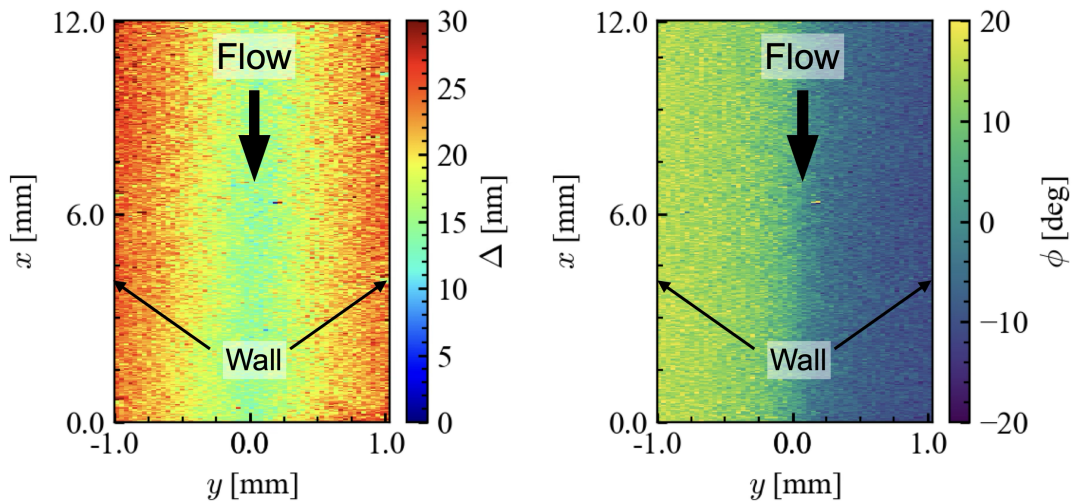


Figure 4. Measurement result of retardation Δ (left) and orientation ϕ (right). The channel aspect ratio is $w/h = 1$. Flow rate is 60 ml/min.

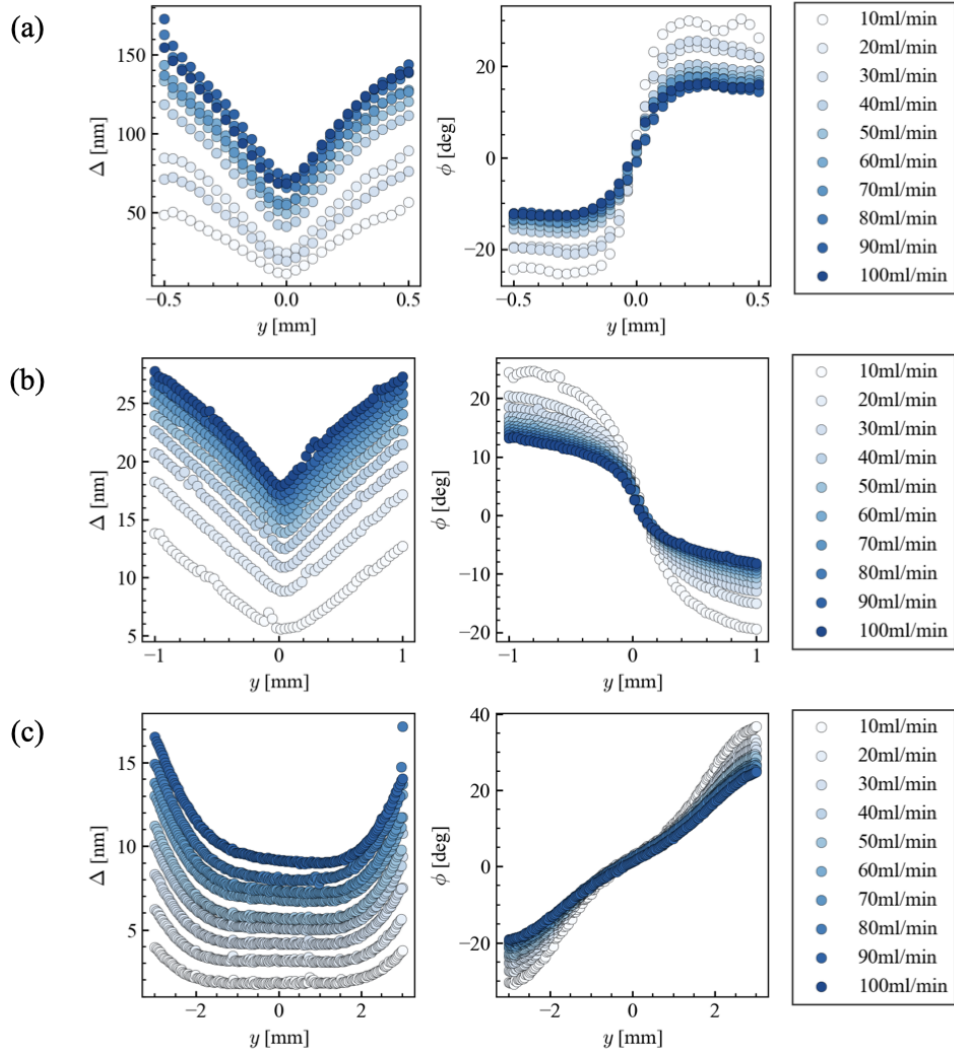


Figure 5. The retardation Δ (left) and orientation ϕ (right), averaged over the time and flow directions, for each channel. (a) is a channel with $w/h = 0.1$, (b) is a channel with $w/h = 1$, (c) is a channel with $w/h = 3$. Flow rate is 10–100 ml/min.

4.2. Differences between SOL and particle orientation-based theory

To separately evaluate the effects of σ_{xy} and σ_{xz} , we analyzed the polarization modulation at the channel wall and the channel center, namely the retardation at the wall, Δ_{wall} , the orientation at the wall, ϕ_{wall} , and the retardation at the channel center, Δ_{center} . In addition, the wall retardation Δ_{wall} and the wall orientation ϕ_{wall} were defined as the absolute averages of the values measured at both channel walls.

Fig. 6 shows the measured results of (a) Δ_{wall} , (b) ϕ_{wall} , and (c) Δ_{center} , together with the corresponding predictions obtained from the stress-optic law (SOL) and the Fokker-Planck equation. The experimental data reported by Nakamine et al. ((Nakamine et al., 2024)) are also included in addition to the results obtained in this study. To compare results obtained for different channel geometries and particle concentrations, Δ_{wall} and Δ_{center} were normalized by Δ_{max} , which is represented by Eq. (8). The polarization modulation based on the Fokker-Planck equation was calculated using Eq. (7) and Eq. (9) to obtain the retardation Δ and the orientation ϕ . The polarization modulation based on SOL was evaluated using Eq. (10) and Eq. (11). In this calculation, the stress-optic coefficients C_1 and C_2 were determined from the results reported by Nakamine et al. (Nakamine et al., 2024): for $w/h = 0.1$, $C_1 = 1.51 \times 10^{-5} \text{ Pa}^{-1}$ and $C_2 = 4.19 \times 10^{-4} \text{ Pa}^{-2}$; for $w/h = 1$, $C_1 = 1.52 \times 10^{-5} \text{ Pa}^{-1}$ and $C_2 = 1.11 \times 10^{-5} \text{ Pa}^{-2}$; and for $w/h = 3$, $C_1 = 1.35 \times 10^{-5} \text{ Pa}^{-1}$. The horizontal axis represents the Peclet number $Pe = |\dot{\gamma}| / D_r$, where the shear rate $|\dot{\gamma}|$ was defined using the second invariant II of the strain-rate tensor $\mathbf{E}$ as $|\dot{\gamma}| = \sqrt{2II}$.

As Peclet number Pe increases, the influence of the flow becomes dominant, causing particles to remain preferentially aligned in the flow direction. In contrast, as Peclet number Pe decreases, the influence of Brownian motion becomes dominant, causing a more random particle orientation distribution.

While the polarization modulation corresponds to a quantity integrated along the camera's optical axis, the Peclet number also varies spatially in the optical axis direction. Therefore, the Peclet number averaged arithmetically along the optical axis Pe_{n1} was introduced as

$$Pe_{n1} = \frac{1}{2h} \int_{-h}^h Pe \, dz. \quad (20)$$

First, we compare the experimental results with the results estimated from SOL. The results predicted by SOL show that, regardless of the value of Pe_{n1} , Δ_{wall} is proportional to the Peclet number Pe_{n1} , and Δ_{center} is proportional to Pe_{n1}^2 . In addition, SOL predicted that the orientation at the channel wall satisfies $\phi_{\text{wall}} = 45^\circ$. In contrast, the experimental results exhibited trends similar to those predicted by SOL in the low-Peclet-number region ($Pe < 1$). However, around $Pe = 1$, deviations from SOL became apparent, and both the retardations, Δ_{wall} and Δ_{center} , as well as the orientation ϕ_{wall} , became smaller than the values predicted by SOL. The current flow field only contains σ_{xy} and σ_{xz} . Therefore, the second-order SOL represented by Eq. (10) and Eq. (11) can be

simplified as follows:

$$\Delta \cos 2\phi = \int_{-h}^h -C_2 \sigma_{xz}^2, dz, \quad (21)$$

$$\Delta \sin 2\phi = \int_{-h}^h 2C_1 \sigma_{xy}, dz. \quad (22)$$

At the channel wall, $\sigma_{xz} = 0$. Therefore, the retardation $\Delta_{\text{wall}}/\Delta_{\text{max}}$ is expected to be proportional to σ_{xy} , and the orientation ϕ_{wall} is expected to satisfy $\phi_{\text{wall}} \simeq 45^\circ$. At the channel center, $\sigma_{xy} = 0$. Therefore, the retardation $\Delta_{\text{center}}/\Delta_{\text{max}}$ is expected to be proportional to σ_{xz}^2 . From the definition of the Peclet number $Pe \propto \eta|\dot{\gamma}|$, the polarization modulation amount estimated from SOL can be expected to be $\Delta_{\text{wall}} \propto Pe$ and $\Delta_{\text{center}} \propto Pe^2$.

However, polarization modulation is not directly determined by the fluid stress itself, but rather by the particle orientation induced by the flow field. As the Peclet number Pe_{n1} increases, particle orientation gradually approaches saturation, causing the linear relationship between the polarization modulation and the fluid stress to break down. Therefore, the experimental results are considered to deviate progressively from the predictions of SOL as the Peclet number increases.

In contrast, predictions based on the Fokker-Planck equations showed good agreement with experimental results. In particular, it can reproduce the tendency for the retardation Δ_{wall} , Δ_{center} , and orientation ϕ_{wall} to become smaller than SOL around $Pe_{n1} = 1$. This is because, in the theory based on the Fokker-Planck equation, the polarization modulation is calculated based on particle orientation, allowing for the estimation of the retardation Δ and orientation ϕ , which take into account the saturation of particle orientation.

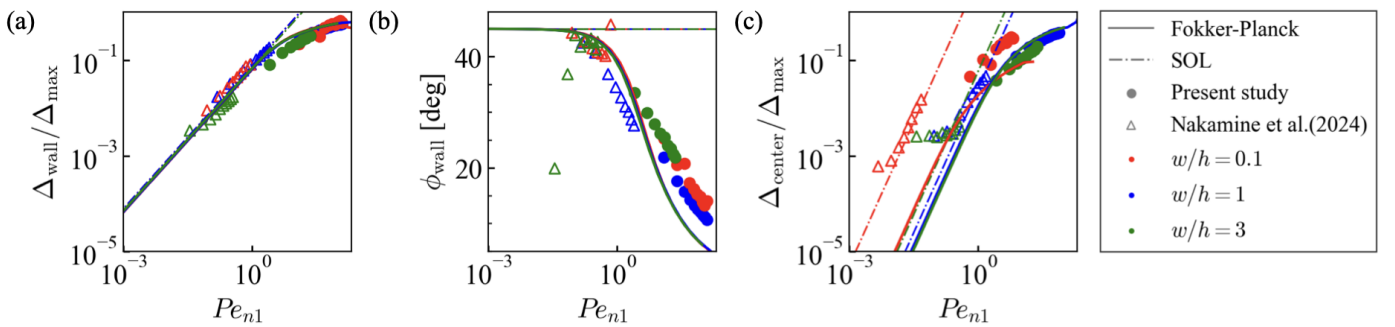


Figure 6. (a) Measurement result of wall's retardation Δ_{wall} . (b) Measurement result of wall's orientation ϕ_{wall} . (c) Measurement result of center's retardation Δ_{center} . The horizontal axis is Pe_{n1} .

4.3. Definition of Pe and its effect on the results

In Fig. 6(c), the results for the channel with $w/h = 0.1$ are plotted at positions shifted from those for the channels with $w/h = 1$ and $w/h = 3$. When comparing the channel geometries, the main

difference specific to the $w/h = 0.1$ channel is the channel depth along the camera's optical axis direction. While the channels with $w/h = 1$ and $w/h = 3$ have a depth of $2h = 2$ mm, the channel with $w/h = 0.1$ has a depth of $2h = 10$ mm. Therefore, it is considered possible that a physical quantity related to the channel depth $2h$ is responsible for this discrepancy. Although several possible factors can be considered, this study focuses on the integration method of the Peclet number Pe .

In Fig. 6, the Peclet number Pe_{n1} defined as the arithmetic mean along the camera optical axis, was used. However, the validity of this definition has not been sufficiently investigated. Therefore, alternative definitions of the Peclet number were examined. The root-mean-square Peclet number, Pe_{n2} , was defined as

$$Pe_{n2} = \sqrt{\frac{1}{2h} \int_{-h}^h Pe^2 dz} \quad (23)$$

and the cubic-mean Peclet number, Pe_{n3} , was defined as

$$Pe_{n3} = \sqrt[3]{\frac{1}{2h} \int_{-h}^h Pe^3 dz}. \quad (24)$$

These definitions were then used to reorganize the experimental results.

The results obtained using the Peclet number Pe_{n2} are shown in Fig. 7, and the results obtained using Pe_{n3} are shown in Fig. 8. Comparison of Fig. 6, Fig. 7, and Fig. 8 indicates that the retardation at the channel wall Δ_{wall} and the orientation at the wall ϕ_{wall} are only weakly dependent on the definition of the Peclet number. In contrast, the phase retardation at the channel center, Δ_{center} , exhibits different trends depending on the definition of Peclet number. In particular, when using Pe_{n2} , the difference due to channel shape became smaller in the low Pe_{n2} region.

This trend was considered based on SOL. At the channel center, the stress-optic coefficient C_2 and the viscosity of the working fluid η of the Newtonian fluid considered in this study are constant. Therefore, Eq. (21) can be rewritten as

$$\begin{aligned} \Delta \cos 2\phi &= \int_{-h}^h -C_2 \sigma_{xz}^2 dz \\ &= -C_2 \eta^2 \int_{-h}^h |\dot{\gamma}|^2 dz \propto \int_{-h}^h |\dot{\gamma}|^2 dz. \end{aligned} \quad (25)$$

Furthermore, because the rotational diffusion coefficient D_r , which appears in the denominator of the Peclet number $Pe = |\dot{\gamma}|/D_r$, is constant for each measurement condition,

$$\begin{aligned} \Delta \cos 2\phi &\propto \int_{-h}^h |\dot{\gamma}|^2 dz \\ &\propto Pe_{n2}^2, \end{aligned} \quad (26)$$

is obtained. Similarly, applying the same derivation to Eq. (25) yields

$$\Delta \sin 2\phi \propto Pe_{n1}. \quad (27)$$

Considering the phase retardation at the channel wall ($\sigma_{xz} = 0$) and at the channel center ($\sigma_{xy} = 0$) in the same manner as discussed previously, the relationships $\Delta_{\text{wall}} \propto Pe_{n1}$ and $\Delta_{\text{center}} \propto Pe_{n2}^2$ are obtained. These results suggest that the arithmetic mean Peclet number Pe_{n1} is appropriate for characterizing the channel wall, whereas the root-mean-square Peclet number Pe_{n2} is more suitable for the channel center.

However, within SOL, the physical interpretation of the second-order term (Terms containing C_2) remains insufficiently understood. Therefore, in order to clarify the physical meaning of this phenomenon, it is necessary to analyze it from the particle orientation, which is the origin of the observed polarization modulation. In particular, in the high Peclet number region where SOL breaks down, discrepancies associated with the channel geometry still remain. To systematically establish this behavior, a redefinition of the Peclet number based on particle orientation theory is required.

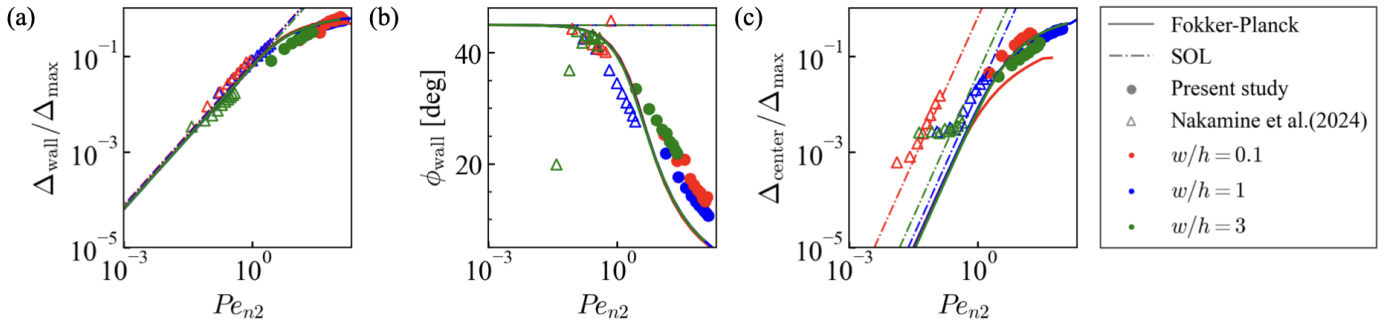


Figure 7. (a) Measurement result of wall's retardation Δ_{wall} . (b) Measurement result of wall's orientation ϕ_{wall} . (c) Measurement result of center's retardation Δ_{center} . The horizontal axis is Pe_{n2} .

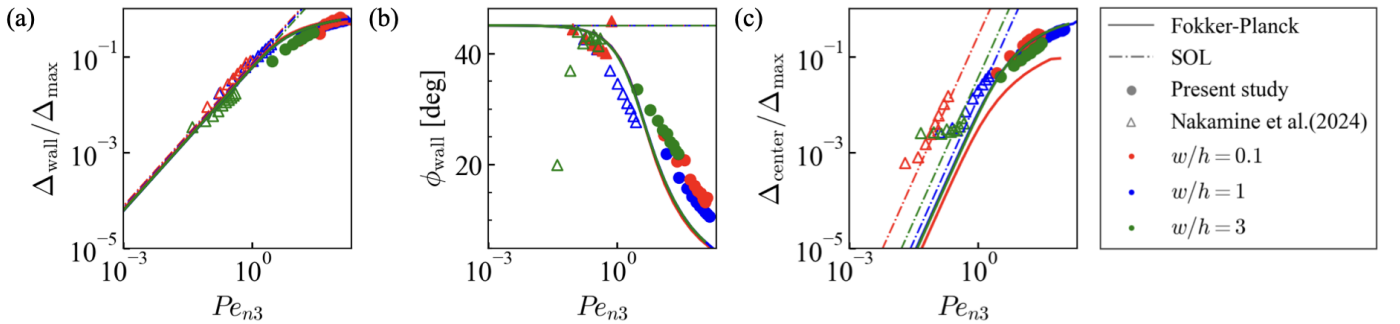


Figure 8. (a) Measurement result of wall's retardation Δ_{wall} . (b) Measurement result of wall's orientation ϕ_{wall} . (c) Measurement result of center's retardation Δ_{center} . The horizontal axis is Pe_{n3} .

5. Conclusion

This study aims to clarify the forward relation between the fluid flow, particle orientation, and polarization modulation, which is essential for stress reconstruction from polarization measurement images. To achieve this objective, we compared the polarization measurement results of steady flow in a rectangular tube with the polarization modulation amount estimated from SOL and the polarization modulation amount estimated from particle orientation theory using the Fokker-Planck equation. Experiments were conducted using three rectangular channels with different aspect ratios, while the flow rate was varied from 10 – 100 mL/min. The SOL employed in this study included second-order stress terms associated with the camera's optical axis direction. The Fokker-Planck equation incorporated particle orientation dynamics based on Jeffery's orbit.

As a result of this comparison, two major findings were obtained. The first finding is that the relationship between polarization modulation and fluid stress depends on the Peclet number, Pe , which represents the ratio between the effect of flow-induced orientation and that of Brownian motion on particle orientation. In the low Peclet number region ($Pe < 1$), the experimental results, SOL predictions, and particle-orientation theory based on the Fokker-Planck equation showed good agreement. However, as the Peclet number increased, the experimental results deviated from SOL's predictions and tended to generally coincide with predictions based on Fokker-Planck theory. This behavior suggests that polarization modulation is not directly determined by the fluid stress itself, but rather originates from particle orientation. In the high Peclet number region, where particle orientation approaches saturation, the linear relationship between fluid stress and particle orientation breaks down, causing the failure of SOL to accurately describe the polarization response. The second finding is that the differences in polarization modulation caused by channel geometry can be established through the definition of the representative Peclet number along the camera's optical axis. Comparisons among the arithmetic mean, root-mean-square, and cubic-mean definitions of the Peclet number showed that the root-mean-square Peclet number is effective for organizing the low Peclet number region at the center of the flow path. On the other hand, discrepancies associated with channel geometry remained in the high Peclet number region. To understand the physical meaning of this phenomenon, further investigation considering the dynamics of particle orientation responsible for the polarization modulation is required. The results obtained in this study provide important insights into the relationship between polarization modulation and fluid stress in flows with spatial distributions along the camera's optical axis.

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