

Irrotational and rotational contributions of material acceleration to unsteady hydrodynamic force

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Unsteady hydrodynamic loading on accelerating bodies can be interpreted as the sum of an irrotational pressure response associated with exchange of kinetic energy with the surrounding fluid and a rotational contribution associated with vorticity generation and viscous diffusion (Lamb, 1932). Force reconstruction is commonly studied by recovering pressure from velocity fields, which is sensitive to differentiation noise and boundary conditions (van Oudheusden, 2013), or by obtaining the drag from the flux of Eulerian vorticity via the Josephson–Anderson relation (Savelli et al., 2026). Thanks to advances in volumetric Lagrangian Particle Tracking, the left-hand side of the incompressible Navier–Stokes momentum equation is directly available from time-resolved trajectories as the material acceleration $\mathbf{a} = D\mathbf{u}/Dt$, enabling a decomposition by applying divergence and curl operators. Using the Helmholtz–Hodge theorem (Caltagirone, 2024), we split $\mathbf{a}$ into a curl-free part $\mathbf{a}_{\text{irr}} = -\nabla\psi$ and a divergence-free (solenoidal) part $\mathbf{a}_{\text{rot}}$. Because the pressure gradient is irrotational, $\mathbf{a}_{\text{irr}}$ isolates the component of $\mathbf{a}$ that determines the instantaneous pressure loading; the associated force captures the added-mass dominated response and remains relevant in viscous and vortical conditions. The complementary solenoidal component $\mathbf{a}_{\text{rot}}$ removes pressure effects and is linked to viscous diffusion and vortex dynamics. The key idea is that applying divergence and curl to Navier–Stokes separates the pressure-controlled (irrotational) and viscous/vortical (rotational) mechanisms (Caltagirone, 2024). We apply this framework to accelerating disc and sphere experiments with force sensors and time-resolved volumetric PTV. We reconstruct F_{irr} and F_{rot} , and compare $F_{\text{total}} = F_{\text{irr}} + F_{\text{rot}}$ to the sensor signal. The results show time-resolved agreement between the reconstructed total force and the sensor signal, while resolving the evolution of irrotational and rotational contributions during vortex formation.

1. Irrotational and rotational split of material acceleration

For incompressible flow in an exterior domain Ω around a rigid body with surface S_b and outward normal $\mathbf{n}$, the momentum equation written in acceleration form is

$$\rho \mathbf{a} = -\nabla p + \mu \nabla^2 \mathbf{u}, \quad \nabla \cdot \mathbf{u} = 0, \quad (1)$$

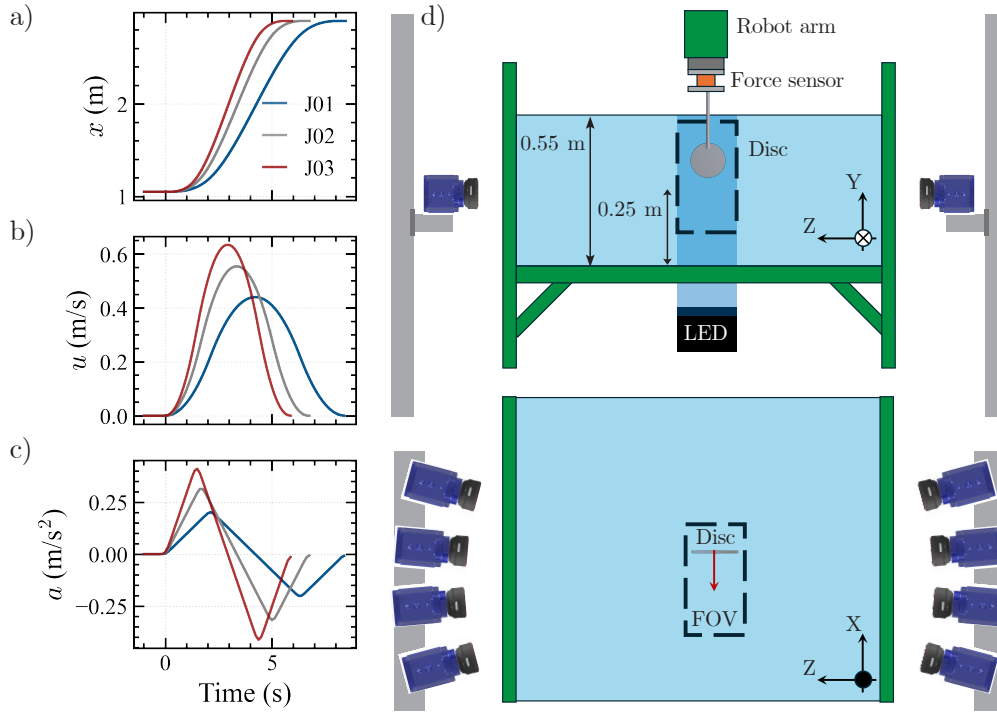


Figure 1. Robot Kinematics, (a) Disc position $x(t)$ for three maximum jerk values. (b) Corresponding disc velocity $u(t)$. (c) Corresponding disc acceleration $a(t)$. (d) Schematic of the experimental setup: the disc is mounted on a robot in a water tank; volumetric illumination is provided by an LED, and the flow is recorded by a multi-camera PTV arrangement. The dashed box indicates the field of view (FOV); axes indicate the laboratory coordinate system.

where $\mathbf{a} = D\mathbf{u}/Dt$ is obtained from time-resolved Lagrangian tracks. Assuming the material acceleration decays at the far-field boundaries, we apply the Helmholtz–Hodge decomposition to the measured $\mathbf{a}$,

$$\mathbf{a} = \mathbf{a}_{\text{irr}} + \mathbf{a}_{\text{rot}}, \quad \mathbf{a}_{\text{irr}} = -\nabla\psi, \quad \nabla \cdot \mathbf{a}_{\text{rot}} = 0, \quad (2)$$

and define the acceleration potential ψ by the standard Poisson problem $\nabla^2\psi = -\nabla \cdot \mathbf{a}$ in Ω with $\partial_n\psi = -\mathbf{a} \cdot \mathbf{n}$ on S_b and decay at the outer boundary. Taking the divergence of (1) gives $\nabla^2 p = -\rho \nabla \cdot \mathbf{a}$, so $p = \rho\psi + C(t)$, i.e. the pressure loading is determined by the curl-free component of $\mathbf{a}$. The corresponding irrotational (pressure/added-mass dominated) force is therefore

$$\mathbf{F}_{\text{irr}} \equiv - \int_{S_b} p \mathbf{n} dS = -\rho \int_{S_b} \psi \mathbf{n} dS. \quad (3)$$

The complementary solenoidal component removes pressure effects. Applying the solenoidal projector to (1) yields $\rho \mathbf{a}_{\text{rot}} = \mu \nabla^2 \mathbf{u}$. Using $\nabla \cdot \boldsymbol{\tau} = \mu \nabla^2 \mathbf{u}$ and neglecting outer-boundary traction contributions, the rotational force, in harmony with the viscous term $\mu \nabla^2 \mathbf{u}$ on the right-hand side of (1), can be written in the volume form

$$\mathbf{F}_{\text{rot}} \equiv \int_{S_b} \boldsymbol{\tau} \mathbf{n} dS = \rho \int_{\Omega} \mathbf{a}_{\text{rot}} dV. \quad (4)$$

Equations (1)–(4) provide a measurement-driven split: $\mathbf{a}_{\text{irr}}$ controls the pressure/added-mass dominated loading, while $\mathbf{a}_{\text{rot}}$ controls the rotational, vortical loading.

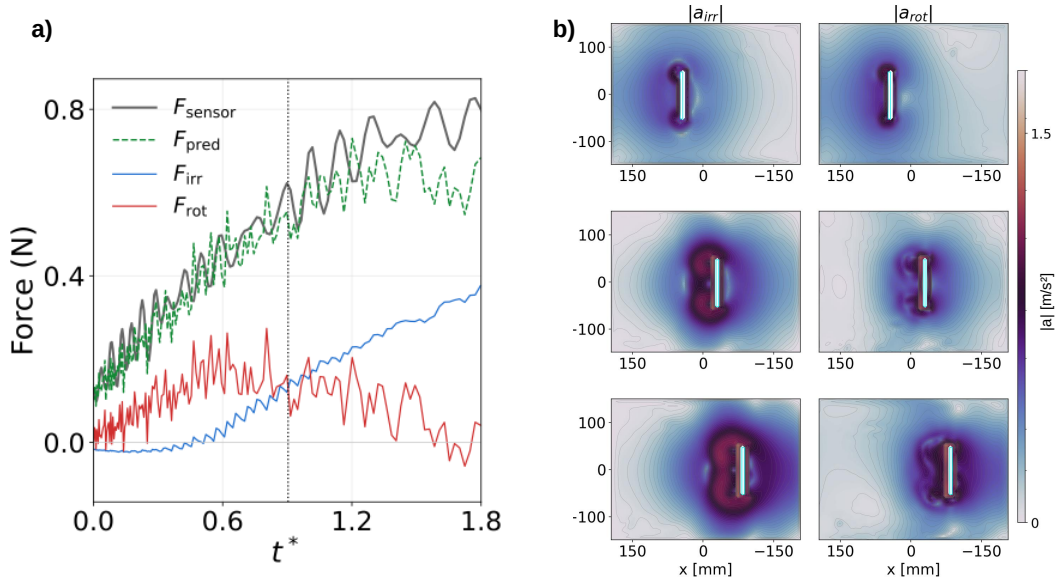


Figure 2. Force reconstruction and material-acceleration field. (a) Force-sensor signal F_{sensor} compared with the HHD-based prediction F_{pred} , together with the irrotational F_{irr} and rotational F_{rot} contributions ($F_{\text{total}} = F_{\text{irr}} + F_{\text{rot}}$), as a function of $t^* = L/D$ (one representative run). Forces are plotted positive in the drag direction; the dotted line marks the crossover where F_{irr} overtakes F_{rot} . (b) Instantaneous magnitude of the irrotational ($|a_{\text{irr}}|$, left) and rotational ($|a_{\text{rot}}|$, right) material acceleration in the symmetry (x - z) plane through the disc at three successive instants (rows); colour denotes $|a|$ in m s^{-2} . The disc (cyan) translates in $-x$, and the near-wake acceleration is associated with vortex-ring formation.

2. Results and discussion

The robot kinematics and the measurement setup are summarised in Fig. 1. The measurement volume was calibrated with a point calibration instead of the usual planar-target calibration. In place of the disc, a single calibration marker was mounted on the robot and moved to a large number of accurately known positions that fill the measurement volume, and each position, read from the robot encoders, provides one reference point seen simultaneously by all cameras. The camera mapping is therefore fitted from a dense three-dimensional cloud of points rather than from a flat pattern imaged at a few discrete depths. This samples the depth continuously and avoids the flatness and alignment errors of a traversed plate; it also ties the calibration and the disc motion to the same robot coordinate system, which removes the registration error between the calibration frame and the kinematic frame. The mapping was then refined by volume self-calibration on the recorded particle images. The volumetric measurement only covered 2.5 seconds of the motion. We normalise time as $t^* = L/D$, where L is the disc displacement from the onset of motion and D its diameter. Figure 2 compares the HHD-based force prediction F_{pred} with the force-sensor data for an accelerating disc, and shows the irrotational F_{irr} and rotational F_{rot} contributions separately. The prediction $F_{\text{pred}} = F_{\text{irr}} + F_{\text{rot}} + F_{\text{inertia}} + F_{\text{strut}}$ adds the body-inertia and strut hydrodynamic corrections to the reconstructed total force, so that it is directly comparable to the sensor. Using the div-curl reformulation of the incompressible Navier-Stokes equation, we separate the irrotational contribution $\mathbf{F}_{\text{irr}}$ obtained from the acceleration potential ψ (pressure/added-mass dominated) and the rotational contribution $\mathbf{F}_{\text{rot}}$ obtained from the solenoidal acceleration (viscous/vortical). Both contributions are significant during vortex-ring formation, i.e. throughout the acceleration

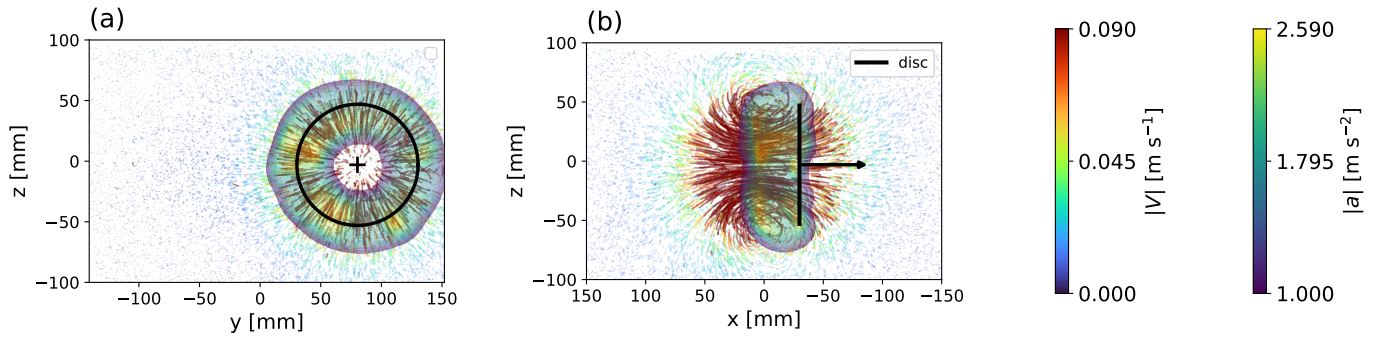


Figure 3. Three-dimensional PTV trajectories and the material acceleration of the vortex ring at $t = 2.00$ s ($t^* = 1.22$), in (a) the y - z plane and (b) the x - z plane. Isosurfaces of the material-acceleration magnitude $|a|$ in the vortex ring; the outer edge is the $|a| = 1 \text{ m s}^{-2}$ isocontour. The disc is drawn in black (the circle and + mark its outline and centre), and the arrow indicates the $-x$ direction of motion.

phase and the vortex ring formation part before the flow transitions to a fully developed wake. The relative weight of the two contributions changes during the motion. In the early stage the rotational contribution F_{rot} is the larger of the two, i.e. vorticity generation and the forming vortex ring carry most of the load. As the disc keeps accelerating, the irrotational contribution F_{irr} grows faster and overtakes F_{rot} at $t^* \approx 0.9$ (dotted line in Fig. 2a), after which the pressure/added-mass channel dominates the loading and F_{rot} falls back toward zero. The same crossover was found in all ten repetitions, and it falls inside the vortex-ring formation stage, so it is a redistribution of the load between the two channels, while the total force stays smooth through the crossover. Figure 3 shows the measured PTV trajectories together with the material-acceleration magnitude $|a|$ at $t = 2.00$ s ($t^* = 1.22$), viewed in the y - z and x - z planes. The strong material acceleration is concentrated in a toroidal tube around the disc rim, in the near wake where the trajectories of the induced vortex-ring flow are fastest, i.e. the region that carries the rotational contribution F_{rot} .

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